INTEGRATION OF IRRATIONAL FUNCTI

Additional standard forms for integration of Irrational functions

The following forms are important for integration of irrational functions

$$\int \sqrt{x^2+a^2}\, dx$$

Put $x = a\sinh\theta$ $\quad dx = a\cosh\theta$

$$I = \int \sqrt{a^2\sinh^2\theta + a^2}\cdot a\cosh\theta\, d\theta$$

$$= a^2 \int \sqrt{\sinh^2\theta + 1}\cdot \cosh\theta\, d\theta$$

$$= a^2 \int \cosh^2\theta\, d\theta \quad (\because \cosh^2\theta - \sinh^2\theta = 1)$$

$$= \frac{a^2}{2}\int (\cosh 2\theta + 1)\, d\theta$$

$$= \frac{a^2}{2}\left[\frac{2\cdot\sinh\theta\cdot\cosh\theta}{2}\right] + \frac{a^2}{2}\sinh^{-1}\frac{x}{a}$$

$$= \frac{a^2}{2} \cdot \frac{x}{a} \cdot \sqrt{1+\sinh^2\theta} + \frac{a^2}{2} \sinh^{-1}\frac{x}{a}$$

$$= \frac{a^2}{2} \cdot \frac{x}{a} \sqrt{1+\frac{x^2}{a^2}} + \frac{a^2}{2} \sinh^{-1}\frac{x}{a}$$

$$= \frac{x \cdot \sqrt{x^2+a^2}}{2} + \frac{a^2}{2} \sinh^{-1}\frac{x}{a}$$

$$= \frac{x\sqrt{x^2+a^2}}{2} + \frac{a^2}{2} \log \frac{x+\sqrt{x^2+a^2}}{a}$$

(ii.) $\int \sqrt{x^2-a^2}\, dx \quad (x > a)$

Put $x = a\cosh\theta \quad \therefore dx = a\sinh\theta\, d\theta$

$$I = \int \sqrt{a^2\cosh^2\theta - a^2} \cdot a\sinh\theta\, d\theta$$

$$= a^2 \int \sinh^2\theta\, d\theta$$

$\because \cosh^2\theta - \sinh^2\theta = 1$

$$= \frac{a^2}{2} \int (\cosh 2\theta - 1)\, d\theta$$

$$= \frac{a^2}{2} \int \cosh 2\theta\, d\theta - \frac{a^2}{2} \int d\theta = \frac{a^2}{2} \frac{\sinh 2\theta}{2}$$

$$- \frac{a^2}{2} \cdot \theta$$

$$\frac{a^2}{2}\cdot\frac{x}{a}\sqrt{\frac{x^2}{a^2}-1}-\frac{a^2}{2}\cdot\cosh^{-1}\frac{x}{a}$$

$$=\frac{x\sqrt{x^2-a^2}}{2}-\frac{a^2}{2}\cdot\log\frac{x+\sqrt{x^2-a^2}}{a}$$

(iii.) $\int\sqrt{a^2-x^2}\,dx \quad (x<a)$

Put $x=a\sin\theta \quad dx=a\cos\theta\,d\theta$

$$I=\int\sqrt{a^2-a^2\sin^2\theta}\cdot a\cos\theta\,d\theta$$

$$=a^2\int\sqrt{1-\sin^2\theta}\cdot\cos\theta\,d\theta=a^2\int\cos^2\theta$$

$$=\frac{a^2}{2}\int(\cos 2\theta+1)\,d\theta=\frac{a^2}{2}\int\cos 2\theta\,d\theta$$

$$+\frac{a^2}{2}\int d\theta.$$

$$\frac{a^2}{2}\cdot\frac{\sin 2\theta}{2}+\frac{a^2}{2}\cdot\theta=\frac{a^2}{2}\cdot\frac{2\sin\theta\cos\theta}{2}$$

$$+\frac{a^2}{2}\sin^{-1}\frac{x}{a}$$

Page No.
Date: / /

$$= \frac{a^2}{2}\cdot\frac{x}{a}\sqrt{1-\sin^2\theta} + \frac{a^2}{2}\cdot\sin^{-1}\frac{x}{a}$$

$$= \frac{ax\cdot\sqrt{a^2-x^2}}{2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}$$